THE THEOREM OF PONTRJAGIN-SCHNIRELMANN AND APPROXIMATE SEQUENCES

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Abstract. In this paper we obtain a modified version of the classical theorem of Pontrjagin and Schnirelmann concerning the covering dimension of a compact metric space and the box-counting dimensions associated with the metrics on the space. T. Miyata and T. Watanabe defined box-counting dimension for approximate sequences $\mathbf{X}$ which represent compact metric spaces X as approximate resolutions $\mathbf{p} : X \to \mathbf{X}$. We show that the covering dimension equals the infimum of the box-counting dimensions associated with the approximate resolutions of the space.

1. Introduction

Let us recall the classical theorem of L. Pontrjagin and L. Schnirelmann [8]:

Theorem. *For any compact metric space X,*

$$\dim X = \inf\{\varliminf_{\varepsilon\to 0}\frac{\log N_{\varepsilon,\mathrm{d}}(X)}{-\log\varepsilon} \;:\; \mathrm{d} \textit{ is a metric on } X\}, \tag{1.1}$$

where $N_{\varepsilon,\mathrm{d}}(X) = \min\{|\mathcal{U}| \;:\; \mathcal{U}$ is a finite open covering of X with mesh $\leq \varepsilon\}$.

Here $|\mathcal{U}|$ denotes the cardinality of $\mathcal{U}$. Note here that $\varliminf_{\varepsilon\to 0}\frac{\log N_{\varepsilon,\mathrm{d}}(X)}{-\log\varepsilon}$ is the lower box-counting dimension of the metric space (X, d).

T. Miyata and T. Watanabe [4, 5, 6] introduced a systematic approach using approximate sequences to study fractal dimensions . The notion of approximate system was first introduced by S. Mardešić and L. Rubin [2]. In order to study the properties of topological spaces, one expands the space X into approximate systems $\mathbf{X}$ and investigates $\mathbf{X}$. T. Miyata and T. Watanabe used the expansions into approximate sequences, called approximate resolutions, to obtain metrics on X and box-counting dimensions of X. So, in view of Pontrjagin and Schnirelmann theorem, a natual question rises: do the box-counting

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dimensions associated with approximate resolutions suffice to get the covering dimension of X?

In [5], it is shown that for each $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ there exist a Cantor set X and a normal approximate resolution $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ of X such that the box-counting dimension of $\mathbf{p}$ equals r. In view of Pontrjagin-Schnirelmann theorem, for Cantor sets X, we observe that the covering dimension $\dim X$ of X is obtained as the infimum of the box-counting dimensions over all normal approximate resolutions of X. Thus the question is: for any compact metric space X, does an equality hold between the covering dimension $\dim X$ of X and the infimum of the box-counting dimensions over all normal approximate resolutions of X?

In this paper we give a positive answer to this question. More precisely, the following is our main theorem:

Theorem. *For any compact metric space X,*

$$\dim X = \inf\{\varliminf_{i\to\infty} \frac{\log_3 \beta_i(\mathbf{X})}{i} \ : \ \mathbf{p} : X \to \mathbf{X} \textit{ is a normal approximate resolution of } X\}.$$

Here for any approximate sequence $\mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$, $\beta_i(\mathbf{X})$ is defined as

$$\beta_i(\mathbf{X}) = \varlimsup_{j\to\infty} N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \text{ for each } i \in \mathbb{N},$$

and $\varliminf_{i\to\infty} \frac{\log_3 \beta_i(\mathbf{X})}{i}$ is the lower box-counting dimension of $\mathbf{p} : X \to \mathbf{X}$.

2. Approximate resolutions

In this section we recall the definition and properties of approximate resolutions which will be needed in later sections. For more details, the reader is referred to [3, 9].

Throughout the paper, *a map means a continuous map.* Let $\mathbb{N}$ denote the usual ordered set of all natural numbers. For any space X, let $\mathrm{Cov}(X)$ denote the set of all open coverings of X. For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$, $\mathcal{U}$ is said to *refine* $\mathcal{V}$, or $\mathcal{U}$ *refines* $\mathcal{V}$, in notation, $\mathcal{U} < \mathcal{V}$, provided for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subseteq V$. For any subset A of X and $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $\mathcal{U}|A = \{U \cap A : U \in \mathcal{U}\}$. If $A = \{x\}$, we write $\mathrm{st}(x, \mathcal{U})$ for $\mathrm{st}(\{x\}, \mathcal{U})$. For each $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}\,\mathcal{U} = \{\mathrm{st}(U, \mathcal{U}) : U \in \mathcal{U}\}$. For any metric space (X, d) and $r > 0$, let $\mathrm{U}_{\mathrm{d}}(x, r) = \{y \in X : \mathrm{d}(x, y) < r\}$. For any subset A of X, let $\mathrm{diam}_{\mathrm{d}}(A)$ denote the diameter of A with respect to the metric

d. For any $\mathcal{U} \in \mathrm{Cov}(X)$, two points $x, x' \in X$ are $\mathcal{U}$-*near*, denoted $(x, x') < \mathcal{U}$, provided $x, x' \in U$ for some $U \in \mathcal{U}$. For any $\mathcal{V} \in \mathrm{Cov}(Y)$, two maps $f, g : X \to Y$ between spaces are $\mathcal{V}$-*near*, denoted $(f, g) < \mathcal{V}$, provided $(f(x), g(x)) < \mathcal{V}$ for each $x \in X$. For each $\mathcal{U} \in \mathrm{Cov}(X)$ and $\mathcal{V} \in \mathrm{Cov}(Y)$, let $f\mathcal{U} = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}\mathcal{V} = \{f^{-1}(V) : V \in \mathcal{V}\}$.

An inverse sequence $(X_i, p_{i,i+1})$ consists of spaces X_i, called *coordinate spaces*, and maps $p_{i,i+1} : X_{i+1} \to X_i$, $i \in \mathbb{N}$. We write p_{ij} for the composite $p_{i,i+1}p_{i+1,i+2} \cdots p_{j-1,j}$ for $i < j$, and let $p_{ii} = 1_{X_i}$, and call the maps p_{ij} *bonding maps*. An *approximate inverse sequence* (*approximate sequence*, in short) $\mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ consists of an inverse sequence $(X_i, p_{i,i+1})$ and $\mathcal{U}_i \in \mathrm{Cov}(X_i)$, $i \in \mathbb{N}$, and must satisfy the following condition:

(AI) For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i_0 > i$ such that $\mathcal{U}_{i'} < p_{ii'}^{-1}\mathcal{U}$ for $i' > i_0$.

An *approximate map* $\mathbf{p} = (p_i) : X \to \mathbf{X}$ of a compact metric space X into an approximate sequence $\mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ consists of maps $p_i : X \to X_i$ for $i \in \mathbb{N}$, called *projection maps*, such that $p_i = p_{ij}p_j$ for $i < j$, and it is an *approximate resolution* of X if it satisfies the following two conditions:

(R1) For each ANR P, $\mathcal{V} \in \mathrm{Cov}(P)$ and map $f : X \to P$, there exist $i \in \mathbb{N}$ and a map $g : X_i \to P$ such that $(gp_i, f) < \mathcal{V}$, and

(R2) For each ANR P and $\mathcal{V} \in \mathrm{Cov}(P)$, there exists $\mathcal{V}' \in \mathrm{Cov}(P)$ such that whenever $i \in \mathbb{N}$ and $g, g' : X_i \to P$ are maps with $(gp_i, g'p_i) < \mathcal{V}'$, then $(gp_{ii'}, g'p_{ii'}) < \mathcal{V}$ for some $i' > i$.

The following is a useful characterization of approximate resolutions:

Theorem 2.1. ([3, Theorem 2.8]) *An approximate map* $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ *is an approximate resolution of* X *if and only if it satisfies the following two conditions:*

(B1) *For each* $\mathcal{U} \in \mathrm{Cov}(X)$, *there exists* $i_0 \in \mathbb{N}$ *such that* $p_i^{-1}\mathcal{U}_i < \mathcal{U}$ *for* $i > i_0$, *and*

(B2) *For each* $i \in \mathbb{N}$ *and* $\mathcal{U} \in \mathrm{Cov}(X_i)$, *there exists* $i_0 > i$ *such that* $p_{ii'}(X_{i'}) \subseteq \mathrm{st}(p_i(X), \mathcal{U})$ *for* $i' > i_0$.

An approximate resolution $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ is said to be *normal* if the family $\{p_i^{-1}\mathcal{U}_i\}$ is a normal sequence on X, i.e., $\mathrm{st}\, p_{i+1}^{-1}\mathcal{U}_{i+1} < p_i^{-1}\mathcal{U}_i$.

Theorem 2.2. ([6, Theorem 2.4]) *Every compact metric space X admits a normal approximate resolution $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ such that all coordinate spaces X_i are finite polyhedra.*

Throughout the paper, *every normal approximate resolution is assumed to have the property of Theorem 2.2.*

Remark 2.3. The notion of approximate system was first introduced by S. Mardešić and L. Rubin [2] in a more general setting. Instead of requiring commutativity $p_{ij}p_{jk} = p_{ik}$ for $i < j < k$, it requires only approximate commutativity. The notion of approximate map in [2] also requires only approximate commutativity instead of requiring commutativity $p_{ij}p_j = p_i$ for $i < j$. S. Mardešić and T. Watanabe [3, 9] extensively studied the approximate resolutions in the most general setting. Although the general setting is required to study various properties of topological spaces, it suffices to use a simpler setting in our case since our primary concern is the class of compact metic spaces.

3. Metrics induced by approximate resolutions

A family $\mathbb{U} = \{\mathcal{U}_i : i \in \mathbb{N}\}$ of open coverings on a space X is said to be a *normal sequence* on X provided $\mathrm{st}\,\mathcal{U}_{i+1} < \mathcal{U}_i$ for each i. In this section, we recall the construction of the metrics induced by approximate resolutions from [4].

Let $\mathbb{U} = \{\mathcal{U}_i : i \in \mathbb{N}\}$ be any normal sequence on a space X with the following property:

(B) For each $x \in X$, $\{\mathrm{st}(x, \mathcal{U}_i) : i \in \mathbb{N}\}$ is a base at x.

Following the approach by P. Alexandroff and P. Urysohn [1] (see also [7]), we define the metric $\mathrm{d}_{\mathbb{U}}$ on X as follows:

$$\mathrm{d}_{\mathbb{U}}(x, x') = \inf\{\mathcal{D}_{\mathbb{U}}(x, x_1) + \mathcal{D}_{\mathbb{U}}(x_1, x_2) + \cdots + \mathcal{D}_{\mathbb{U}}(x_n, x')\}$$

where the infimum is taken over all points $x_1, x_2, ..., x_n$ in X, and

$$\mathcal{D}_{\mathbb{U}}(y, z) = \begin{cases} 9 & \text{if } (y, z) \not< \mathcal{U}_1; \\ \dfrac{1}{3^{i-2}} & \text{if } (y, z) < \mathcal{U}_i \text{ but } (y, z) \not< \mathcal{U}_{i+1}\ ; \\ 0 & \text{if } (y, z) < \mathcal{U}_i \text{ for all } i \in \mathbb{N}\ . \end{cases}$$

Proposition 3.1. ([4, Proposition 3.1]) *The metric $\mathrm{d}_{\mathbb{U}}$ has the property*

$$\mathrm{st}(x, \mathcal{U}_{i+3}) \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{U}}}(x, \frac{1}{3^i}) \subseteq \mathrm{st}(x, \mathcal{U}_i) \text{ for each } x \in X \text{ and } i \in \mathbb{N}. \tag{3.1}$$

In particular, if $\mathbb{B} = \{\mathcal{B}_i : i \in \mathbb{N}\}$ is the normal sequence on a metric space (X, d) such that $\mathcal{B}_i = \{\mathrm{U}_\mathrm{d}(x, \frac{1}{3^i}) : x \in X\}$, then the metric $\mathrm{d}_\mathbb{B}$ induces a uniformity which is equivalent to that induced by the metric d. Moreover, it is proven in [4, Proposition 3.7] that if X is a convex subset of a normed linear space, and if X is equipped with the metric d which is induced by the norm, then there is a constant $c > 0$ such that $d_\mathbb{U}(x, x') = c\,\mathrm{d}(x, x')$ for some $x, x' \in X$.

Throughout the rest of the paper, *every normal sequence is assumed to have property* (B).

Let $\mathbf{p} : X \to \mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ be any normal approximate resolution of X. Then for any $x, x' \in X$, we define the function $\mathcal{D}_\mathbf{p} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_\mathbf{p}(x, x') = \begin{cases} 9 & \text{if } (p_i(x), p_i(x')) \not< \mathcal{U}_i \text{ for any } i; \\ \frac{1}{3^{i-2}} & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ but } (p_i(x), p_i(x')) \not< \mathcal{U}_{i+1} \ ; \\ 0 & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ for all } i \ , \end{cases}$$

and the function $\mathrm{d}_\mathbf{p} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_\mathbf{p}(x, x') = \inf\{\mathcal{D}_\mathbf{p}(x, x_1) + \mathcal{D}_\mathbf{p}(x_1, x_2) + \cdots + \mathcal{D}_\mathbf{p}(x_n, x')\}$$

where the infimum is taken over all finitely many points $x_1, x_2, ..., x_n$ of X. Then property (B1) implies that the normal sequence $\{p_i^{-1}\mathcal{U}_i\}$ has property (B), and the function $\mathrm{d}_\mathbf{p}$ is a metric on X. Indeed, $\mathrm{d}_\mathbf{p}(x, x') = \mathrm{d}_\mathbb{U}(x, x')$ for any $x, x' \in X$, wehre $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$.

4. Main result

In this section we prove the main theorem of the paper and some consequences of the result.

For any covering $\mathcal{U}$ of a compact metric space X, let $N_\mathcal{U}(X) = \min\{n : X \subseteq U_1 \cup \cdots \cup U_n, U_i \in \mathcal{U}\}$.

Lemma 4.1. *Let $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ be an approximate resolution of X. Then we have the following properties:*

(a) *For every $i \in \mathbb{N}$ there exists $i_0 \geq i$ such that $N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \leq N_{p_i^{-1}\mathcal{U}_i}(X)$ for $j \geq i_0$.*

(b) *$N_{p_i^{-1}\mathcal{U}_i}(X) \leq N_{p_{ij}^{-1}\mathcal{U}_i}(X_j)$ for $i \leq j$.*

Proof. (a) is proven in [5, Proposition 5.1]. For (b), let $i \leq j$, and suppose $n = N_{p_{ij}^{-1}\mathcal{U}_i}(X_j)$. Then there exist $U_1, \ldots, U_n \in \mathcal{U}_j$ such that $X_j = p_{ij}^{-1}(U_1) \cup \cdots \cup p_{ij}^{-1}(U_n)$. So, $X =$

$p_j^{-1}p_{ij}^{-1}(U_1) \cup \cdots \cup p_j^{-1}p_{ij}^{-1}(U_n)$. Since $p_{ij}p_j = p_i$, $X = p_i^{-1}(U_1) \cup \cdots \cup p_i^{-1}(U_n)$. Thus $N_{p_i^{-1}\mathcal{U}_i}(X) \leq n$, as required. $\square$

For any approximate sequence $\mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$, $\beta_i(\mathbf{X})$ is defined as

$$\beta_i(\mathbf{X}) = \overline{\lim_{j\to\infty}} N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \text{ for each } i \in \mathbb{N}.$$

Then the lower box-counting dimension $\underline{\dim}_B(\mathbf{p})$ of $\mathbf{p} : X \to \mathbf{X}$ is defined as $\underline{\lim}_{i\to\infty} \dfrac{\log_3 \beta_i(\mathbf{X})}{i}$ [5].

Theorem 4.2. *For any compact metric space X,*

$$\dim X = \inf\{\underline{\lim}_{i\to\infty} \frac{\log_3 \beta_i(\mathbf{X})}{i} \;:\; \mathbf{p} : X \to \mathbf{X} \textit{ is a normal approximate resolution of } X\},$$

Proof. It suffices to show

$$\begin{aligned}&\inf\{\underline{\lim}_{\varepsilon\to 0} \frac{\log N_{\varepsilon,\mathrm{d}}(X)}{-\log\varepsilon} \;:\; \mathrm{d} \text{ is a metric on } X\}\\ &= \inf\{\underline{\lim}_{i\to\infty} \frac{\log_3 \beta_i(\mathbf{X})}{i} \;:\; \mathbf{p} : X \to \mathbf{X} \text{ is a normal approximate resolution of } X\}\end{aligned} \tag{4.1}$$

First, let $\mathbf{p} = (p_i) : X \to \mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ be a normal approximate resolution of X. Consider the normal sequence $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$. By Lemma 4.1, for each $i \in \mathbb{N}$, $\beta_i(\mathbf{X}) = \overline{\lim}_{j\to\infty} N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) = N_{p_i^{-1}\mathcal{U}_i}(X)$. So,

$$\underline{\lim}_{i\to\infty} \frac{\log_3 \beta_i(X)}{i} = \underline{\lim}_{i\to\infty} \frac{\log_3 N_{p_i^{-1}\mathcal{U}_i}(X)}{i}. \tag{4.2}$$

Consider the metric $\mathrm{d}_{\mathbf{p}}$ induced by the approximate resolution $\mathbf{p}$. Let $i > 3$, and suppose that $X \subseteq p_i^{-1}(U_1) \cup \cdots \cup p_i^{-1}(U_n)$ for some $U_1, \ldots, U_n \in \mathcal{U}_i$. Then by Proposition 3.1, for each $j = 1, \ldots, n$, if $x \in U_j$, $p_i^{-1}(U_j) \subseteq \mathrm{st}(x, p_i^{-1}\mathcal{U}_i) \subseteq U_j' = \mathrm{U}_{\mathrm{d}_{\mathbf{p}}}(x, \frac{1}{3^{i-3}})$. But $\mathrm{diam}_{\mathrm{d}_{\mathbf{p}}}(U_j') \leq \frac{2}{3^{i-3}} < \frac{1}{3^{i-2}}$, and hence $N_{p_i^{-1}\mathcal{U}_i}(X) \geq N_{\frac{1}{3^{i-2}},\mathrm{d}_{\mathbb{U}}}(X)$. This implies

$$\frac{\log_3 N_{p_i^{-1}\mathcal{U}_i}(X)}{i} \geq \frac{\log N_{\frac{1}{3^{i-2}},\mathrm{d}_{\mathbf{p}}}(X)}{-\log\frac{1}{3^i}} = \frac{\log N_{\frac{1}{3^{i-2}},\mathrm{d}_{\mathbf{p}}}(X)}{-\log\frac{1}{3^{i-2}} + \log 3^2},$$

and hence

$$\underline{\lim}_{i\to\infty} \frac{\log_3 N_{p_i^{-1}\mathcal{U}_i}(X)}{i} \geq \underline{\lim}_{i\to\infty} \frac{\log N_{\frac{1}{3^i},\mathrm{d}_{\mathbf{p}}}(X)}{-\log\frac{1}{3^i}}.$$

The right hand side of the inequality equals $\underline{\lim}_{\varepsilon\to 0} \dfrac{\log N_{\varepsilon,\mathrm{d}_{\mathbf{p}}}(X)}{-\log\varepsilon}$. This together with (4.2) implies the inequality " $\leq$ " in (4.1).

To show the reverse inequality in (4.1), let d be a metric on X. Let $\mathcal{U}$ be a finite open covering of X with mesh $\leq \frac{1}{3^i}$. Consider the normal sequence $\mathbb{B} = \{\mathcal{B}_i\}$ consisting of the open coverings $\mathcal{B}_i = \{\mathrm{U}_{\mathrm{d}}(x, \frac{1}{3^i}) : x \in X\}$. If $X \subseteq U_1 \cup \cdots \cup U_n$ for some $U_1, \ldots, U_n \in \mathcal{U}$, then $X \subseteq B_1 \cup \cdots \cup B_n$ for some $B_1, \ldots, B_n \in \mathcal{B}_i$. So, $N_{\frac{1}{3^i},\mathrm{d}}(X) \geq N_{\mathcal{B}_i}(X)$. This implies

$$\underline{\lim}_{i\to\infty} \frac{\log N_{\frac{1}{3^i},\mathrm{d}}(X)}{-\log \dfrac{1}{3^i}} \geq \underline{\lim}_{i\to\infty} \frac{\log_3 N_{\mathcal{B}_i}(X)}{i}. \tag{4.3}$$

There is an isometrical embedding of X into a normed linear space L. Consider X as a closed subset of its convex hull K. For each $i \in \mathbb{N}$ there is an isomorphic extension $\mathcal{U}'_i = \{U_\alpha\}$ of $\mathcal{B}_i = \{B_\alpha\}$ over some neighborhood of X in K so that U_α is contained in the $\frac{1}{i}$-neighborhood of B_α. Choose a sequence of compact polyhedral neighborhoods X_i of X such that $X_{i+1} \subseteq X_i$ and $X_i \subseteq \cup\, \mathcal{U}'_i$. For each $i \in \mathbb{N}$, let $\mathcal{U}_i = \mathcal{U}'_i | X_i$, and let $p_{i,i+1} : X_{i+1} \to X_i$ be the inclusion map.

We show that $\mathbf{X} = (X_i, \mathcal{U}_i, p_{i,i+1})$ is an approximate sequence. To see this, we must verify property (AI). Let $i \in \mathbb{N}$, and let $\mathcal{U}$ be an open covering of X_i. Without loss of generality, we can assume that $\mathcal{U}$ is a finite open covering. Let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{U}$. Choose $i_0 \geq i$ so that $\frac{1}{i_0} < \frac{\varepsilon}{4}$. So, $\frac{1}{3^{i_0}} < \frac{\varepsilon}{4}$. Then we have $\mathcal{U}_j < p_{ij}^{-1}\mathcal{U} = \mathcal{U}|X_j$ for $j \geq i_0$. Indeed, if $U \in \mathcal{U}_j$, then U is contained in the $\frac{1}{j}$-neighborhood of some $B \in \mathcal{B}_j$, so $\mathrm{diam}_{\mathrm{d}}(U) < \frac{2}{3^j} + \frac{2}{j} < \varepsilon$. Since ε is a Lebesgue number of $\mathcal{U}$, U is contained in some open set from $\mathcal{U}$.

For each $i \in \mathbb{N}$, let $p_i : X \to X_i$ be the inclusion map. Then $\mathbf{p} = (p_i) : X \to \mathbf{X}$ is a normal approximate resolution of X. To see this, it is enough to verify properties (B1) and (B2) in Theorem 2.1. Indeed, we can verify property (B1) in the same way as for property (AI), and property (B2) immediately follows from the definition of X_i. That the approximate resolution is normal follows from the fact that $\mathbb{B}$ is a normal sequence.

Moreover, there is $i_1 \in \mathbb{N}$ such that $N_{\mathcal{B}_i}(X) = N_{p_i^{-1}\mathcal{U}_i}(X) = \beta_i(\mathbf{X})$ for $i \geq i_1$ by Lemma 4.1, and hence

$$\underline{\lim}_{i\to\infty} \frac{\log_3 N_{\mathcal{B}_i}(X)}{i} = \underline{\lim}_{i\to\infty} \frac{\log_3 \beta_i(\mathbf{X})}{i}.$$

This together with (4.3) implies the inequality "$\geq$" in (4.1). This completes the proof of the theorem. $\square$

Remark 4.3. In a similar way, using [5, Proposition 5.1] instead of Lemma 4.1, we can see that Theorem 4.2 still holds if we replace our notion of approximate resolution by that in the sense of [3], i.e., the almost commutative version of approximate resolution.

By the first part of the proof of Theorem 4.2 we have

Corollary 4.4. *For any compact metric space X,*

$$\dim X = \inf\{\varliminf_{\varepsilon\to 0}\frac{\log N_{\varepsilon,\mathrm{d}_{\mathbf{p}}}(X)}{-\log\varepsilon} \ : \ \mathbf{p}: X\to\mathbf{X} \textit{ is a normal approximate resolution of } X\}.$$

By the second part and an easy modification of the first part of the proof of Theorem 4.2 we have

Corollary 4.5. *For any compact metric space X,*

$$\dim X = \inf\{\varliminf_{i\to\infty}\frac{\log_3 N_{\mathcal{U}_i}(X)}{i} \ : \ \mathbb{U}=\{\mathcal{U}_i\} \textit{ is a normal sequence on } X\}.$$

For any compact metric spaces X and Y with normal sequences $\mathbb{U}$ and $\mathbb{V}$, respectively, a map $f : X \to Y$ is a $(\mathbb{U},\mathbb{V})$-Lipschitz map provided there is a constant $c > 0$ such that

$$\mathrm{d}_{\mathbb{V}}(f(x), f(y)) \le c\,\mathrm{d}_{\mathbb{U}}(x, y) \text{ for } x, y \in X.$$

The $(\mathbb{U},\mathbb{V})$-Lipschitz maps are characterized by a property of the normal sequeces $\mathbb{U}$ and $\mathbb{V}$.

Theorem 4.6. ([4, Theorem 5.2]) *Let X and Y be compact metric spaces with normal sequences $\mathbb{U}=\{\mathcal{U}_i\}$ and $\mathbb{V}=\{\mathcal{V}_i\}$, respectively, and let $f : X \to Y$ be a map. For $m \in \mathbb{N}\cup\{0\}$ consider the following conditions:*

(L)$_m$: $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \le 3^m\,\mathrm{d}_{\mathbb{U}}(x, x')$ *for* $x, x' \in X$,

(M)$_m$: $\mathcal{V}_{i+n} < f^{-1}\mathcal{V}_i$ *for* $i \in \mathbb{N}$.

Then the following implications hold:

(a) *(M)$_m$ $\Rightarrow$ (L)$_m$,*

(b) *(L)$_m$ $\Rightarrow$ (M)$_{m+4}$*

As a consequence of Corollary 4.5 there is a result relating $(\mathbb{U},\mathbb{V})$-Lipschitz maps to the covering dimension.

Corollary 4.7. *For any surjective map $f : X \to Y$ between compact metric spaces,*

(a) *if every normal sequence* $\mathbb{U} = \{\mathcal{U}_i\}$ *on* X *admits a normal sequence* $\mathbb{V} = \{\mathcal{V}_i\}$ *on* Y *with property (M)*$_m$ *for some* $m \in \mathbb{N} \cup \{0\}$*, then* $\dim X \geq \dim Y$*.*

(b) *if every normal sequence* $\mathbb{U} = \{\mathcal{U}_i\}$ *on* X *admits a normal sequence* $\mathbb{V} = \{\mathcal{V}_i\}$ *on* Y *such that there is* $k \in \mathbb{N}$ *with* $\mathcal{U}_i < f^{-1}\mathcal{V}_{ki}$ *for* $i \in \mathbb{N}$*, then* $\dim X \geq k \dim Y$*.*

Proof. The two assertions follow from Corollary 4.5 and the facts that

$$\frac{\log_3 N_{\mathcal{U}_i}(X)}{i} \geq \frac{\log_3 N_{\mathcal{V}_{i-m}}(X)}{i}$$

for (a) and that

$$\frac{\log_3 N_{\mathcal{U}_i}(X)}{i} \geq \frac{\log_3 N_{\mathcal{V}_{ki}}(X)}{ki} \cdot k$$

for (b). □

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